

Exercise 24

Solve the differential equation using the method of variation of parameters.

$$y'' + y = \sec^3 x, \quad 0 < x < \pi/2$$

Solution

Since the ODE is linear, the general solution can be written as the sum of a complementary solution and a particular solution.

$$y = y_c + y_p$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' + y_c = 0 \tag{1}$$

This is a linear homogeneous ODE, so its solutions are of the form $y_c = e^{rx}$.

$$y_c = e^{rx} \quad \rightarrow \quad y_c' = r e^{rx} \quad \rightarrow \quad y_c'' = r^2 e^{rx}$$

Plug these formulas into equation (1).

$$r^2 e^{rx} + e^{rx} = 0$$

Divide both sides by e^{rx} .

$$r^2 + 1 = 0$$

Solve for r .

$$r = \{-i, i\}$$

Two solutions to the ODE are e^{-ix} and e^{ix} . By the principle of superposition, then,

$$\begin{aligned} y_c(x) &= C_1 e^{-ix} + C_2 e^{ix} \\ &= C_1(\cos x - i \sin x) + C_2(\cos x + i \sin x) \\ &= (C_1 + C_2) \cos x + (-iC_1 + iC_2) \sin x \\ &= C_3 \cos x + C_4 \sin x. \end{aligned}$$

On the other hand, the particular solution satisfies the original ODE.

$$y_p'' + y_p = \sec^3 x \tag{2}$$

In order to obtain a particular solution, use the method of variation of parameters: Allow the parameters in the complementary solution to vary.

$$y_p = C_3(x) \cos x + C_4(x) \sin x$$

Differentiate it with respect to x .

$$y_p' = C_3'(x) \cos x + C_4'(x) \sin x - C_3(x) \sin x + C_4(x) \cos x$$

If we set

$$C_3'(x) \cos x + C_4'(x) \sin x = 0, \tag{3}$$

then

$$y'_p = -C_3(x) \sin x + C_4(x) \cos x.$$

Differentiate it with respect to x once more.

$$y''_p = -C'_3(x) \sin x + C'_4(x) \cos x - C_3(x) \cos x - C_4(x) \sin x$$

Substitute these formulas into equation (2).

$$[-C'_3(x) \sin x + C'_4(x) \cos x - \cancel{C_3(x) \cos x} - \cancel{C_4(x) \sin x}] + [\cancel{C_3(x) \cos x} + \cancel{C_4(x) \sin x}] = \sec^3 x$$

Simplify the result.

$$-C'_3(x) \sin x + C'_4(x) \cos x = \sec^3 x \quad (4)$$

Multiply both sides of equation (3) by $\sin x$, and multiply both sides of equation (4) by $\cos x$.

$$\begin{aligned} C'_3(x) \cos x \sin x + C'_4(x) \sin^2 x &= 0 \\ -C'_3(x) \cos x \sin x + C'_4(x) \cos^2 x &= \sec^2 x \end{aligned}$$

Add the respective sides of these equations to eliminate $C'_3(x)$.

$$C'_4(x) = \sec^2 x$$

Integrate this result to get $C_4(x)$, setting the integration constant to zero.

$$C_4(x) = \tan x$$

Multiply both sides of equation (3) by $\cos x$, and multiply both sides of equation (4) by $-\sin x$.

$$\begin{aligned} C'_3(x) \cos^2 x + C'_4(x) \cos x \sin x &= 0 \\ C'_3(x) \sin^2 x - C'_4(x) \cos x \sin x &= -\sec^3 x \sin x \end{aligned}$$

Add the respective sides of these equations to eliminate $C'_4(x)$.

$$C'_3(x) = -\sec^3 x \sin x$$

Integrate this result to get $C_3(x)$, setting the integration constant to zero.

$$\begin{aligned} C_3(x) &= \int^x C'_3(w) dw \\ &= -\int^x \frac{\sin w}{\cos^3 w} dw \\ &= \int^{\cos x} \frac{du}{u^3} \\ &= \left(-\frac{1}{2u^2} \right) \Big|_{\cos x} \\ &= -\frac{1}{2 \cos^2 x} \end{aligned}$$

Therefore,

$$\begin{aligned}y_p &= C_3(x) \cos x + C_4(x) \sin x \\&= \left(-\frac{1}{2 \cos^2 x}\right) \cos x + (\tan x) \sin x \\&= -\frac{1}{2 \cos x} + \frac{\sin^2 x}{\cos x} \\&= \frac{-1 + 2 \sin^2 x}{2 \cos x} \\&= -\frac{\cos 2x}{2 \cos x},\end{aligned}$$

and the general solution to the ODE is

$$\begin{aligned}y(x) &= y_c + y_p \\&= C_3 \cos x + C_4 \sin x - \frac{\cos 2x}{2 \cos x},\end{aligned}$$

where C_3 and C_4 are arbitrary constants.